Math 53: Multivariable Calculus **Sections 102, 103**

Handout for 2020-02-12

Conceptual questions

Question 1. Starting with $\mathbf{r}$, how do you compute $\mathbf{r}'$, $\mathbf{r}''$, $\mathbf{T}$, $\mathbf{N}$?

Question 2. Let a be a constant. Consider the curve

$$\mathbf{r}(t) = \langle at, 3\cos t, 3\sin t\rangle.$$

Without doing any calculations:

- What does this curve look like? Do you know its name?
- What is its (exact) curvature if $a = 0$? Yesterday you computed the curvature when $a = 1$. What happens to its curvature if you let $a \to \infty$ or $a \to -\infty$?

Question 3. Suppose a particle is moving with constant speed, with position given by $\mathbf{r}(t)$. Why does this *not* imply $\mathbf{r}''(t) = \mathbf{0}$?

Computations

Problem 1. Show that the curves $\mathbf{r}_1(t) = \langle 2t, 2-2t, 3+t^2\rangle$ and $\mathbf{r}_2(s) = \langle 6-2s, 2s-4, s^2\rangle$ intersect perpendicularly.
(You should first verify that they intersect at all!)

Problem 2. In the third conceptual question above, you saw that motion with constant speed does not imply $\mathbf{r}'' = \mathbf{0}$. However, it does imply that $\mathbf{r}'$ and $\mathbf{r}''$ are orthogonal at all times. Can you prove this?
Hint: Philip showed something very similar in lecture.

Problem 3. Suppose that $\mathbf{r}(0) = \langle 1, -2, 1\rangle$ and $\mathbf{r}'(t)$ is always orthogonal to $\langle 5, 2, 2\rangle$. What can you say about the curve that $\mathbf{r}(t)$ describes?

Problem 4. Let C be the curve of intersection of the surfaces $y = x^2$ and $z = x^3$. Find the radius and center of the osculating circle to the curve C at the point $(0, 0, 0)$.

Question 2.

- It is a helix when $a \neq 0$, and a circle when $a = 0$.
- When $a = 0$, the curvature is the reciprocal of the radius of the circle: 1/3. You saw in lecture that $\kappa = 0.3$ when $a = 1$. As $a \to \pm\infty$, the curvature tends to zero.

Question 3. For example, circular motion $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ has constant speed but nonzero acceleration vector.

Problem 1. The system of equations

$$\begin{aligned} 2t &= 6 - 2s \\ 2 - 2t &= 2s - 4 \\ 3 + t^2 &= s^2 \end{aligned}$$

has the single solution $t = 1, s = 2$. Then check that $\mathbf{r}_1'(1) \cdot \mathbf{r}_2'(2) = 0$.

Problem 2. Differentiate the equation $|\mathbf{r}'(t)|^2 = \text{constant}$ with respect to t, writing the left hand side as a dot product and using the product rule.

Problem 3. The curve is entirely contained in the plane $5(x-1) + 2(y+2) + 2(z-1) = 0$.

Problem 4. Parametrize the curve: $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ will do. One finds that $\mathbf{N}(0) = \langle 0, 1, 0 \rangle$ and $\kappa(0) = 2$. Thus the radius of the osculating circle is 1/2 and it has center

$$\langle 0, 0, 0 \rangle + \frac{1}{2}\langle 0, 1, 0 \rangle = \langle 0, 1/2, 0 \rangle$$

i.e. the point $(0, 1/2, 0)$. It is contained in the xy-plane.